Principal values of some integral functionals of fractional Brownian motion

Litan Yan (Donghua University)
(A joint work with Xichao Sun and Le Wang)

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The results in this talk come from the following manuscript:

- (with X. Sun and L. Wang) Principal values of some integral functionals of fractional Brownian motion, preprint 2019.

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- In this talk, we consider the existence of the limit in L^2 (resp. almost surely)

$$\mathcal{K}_t^{H,f} := \lim_{\varepsilon\downarrow 0}\left(\int_0^t f(B_s^H)1_{\{|B_s^H|>\varepsilon\}}ds^{2H} + \zeta_t^H(\varepsilon)\right), \quad t\geq 0,$$

where B^H is a fractional Brownian motion with Hurst index $0 < H < 1$, and f is not locally integrable, i.e., the integral

$$\int_{-M}^{M}|f(x)|dx = \infty \quad (\exists M > 0).$$

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where B^H is a fractional Brownian motion with Hurst index $0 < H < 1$, and f is not locally integrable, i.e., the integral

$$\int_{-M}^{M}|f(x)|dx = \infty \quad (\exists M > 0).$$

- The term $\zeta_t^H(\varepsilon)$ is defined as follows

$$\zeta_t^H(\varepsilon) := \mathscr{L}^H(\varepsilon,t)g(\varepsilon) - \mathscr{L}^H(-\varepsilon,t)g(-\varepsilon),$$

where g is the primitive function of f and $\mathscr{L}^H(x,t) = \int_0^t \delta(B_s^H - x)ds^{2H}$ is the weighted local time of B^H.

- It is important to note that if the limit

$$\lim_{\varepsilon \downarrow 0} \zeta_t^H(\varepsilon)$$

exists in probability, we usually call the limit (in probability)

$$\lim_{\varepsilon \downarrow 0} \int_0^t f(B_s^H) 1_{\{|B_s^H| > \varepsilon\}} ds^{2H}$$

the (Cauchy) principal value of the integral $\int_0^t f(B_s^H) ds^{2H}$, and it also is denoted by

$$\text{p.v.} \int_0^t f(B_s^H) ds^{2H}.$$

Example (1)

Let $f(x) = \frac{1}{x}$. We have that $g(x) = \log x$,

$$\lim_{\varepsilon \downarrow 0} \zeta_t^H(\varepsilon) = 0$$

in L^2 and almost surely and

$$\text{p.v.} \int_0^t \frac{1}{B_s^H} ds^{2H} = \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|B_s^H| > \varepsilon\}} \frac{1}{B_s^H} ds^{2H}$$

in L^2 and almost surely.

Example (2)

Let $f(x) = \frac{1}{x} - \mathrm{sign}(x)\sin x$. We then have

$$\lim_{\varepsilon\downarrow 0} \zeta_t^H(\varepsilon) = 2\mathscr{L}^H(0,t)$$

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$$\text{p.v.} \int_0^t \frac{1}{B_s^H} ds^{2H} = \lim_{\varepsilon\downarrow 0} \int_0^t 1_{\{|B_s^H|>\varepsilon\}} f(B_s^H) ds^{2H}$$

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- We now consider the conditions which the limit $\lim_{\varepsilon\downarrow 0} \zeta_t^H(\varepsilon)$ exists in probability.

- Consider the decomposition

$$
\begin{aligned}
\zeta_t^H(\varepsilon) &= \mathscr{L}^H(\varepsilon,t)g(\varepsilon) - \mathscr{L}^H(-\varepsilon,t)g(-\varepsilon) \\
&= \mathscr{L}^H(\varepsilon,t)\Big[g(\varepsilon) - g(-\varepsilon)\Big] \\
&\qquad + g(-\varepsilon)\Big[\mathscr{L}^H(\varepsilon,t) - \mathscr{L}^H(-\varepsilon,t)\Big]
\end{aligned}
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  for all $t \geq 0$ and $\varepsilon > 0$.
- By the continuity of $x \mapsto \mathscr{L}^H(x,t)$, when the function $x \mapsto g(x)$ does not increase too fast at $x = 0$, for example, $x^{\frac{1-H}{2H}} g(x) \to 0$ (as $x \downarrow 0$) we can get that

$$g(-\varepsilon)\Big[\mathscr{L}^H(\varepsilon,t) - \mathscr{L}^H(-\varepsilon,t)\Big] \longrightarrow 0 \quad (\varepsilon \downarrow 0)$$

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$$g(-\varepsilon)\Big[\mathscr{L}^H(\varepsilon,t) - \mathscr{L}^H(-\varepsilon,t)\Big] \longrightarrow 0 \quad (\varepsilon \downarrow 0)$$

  in L^2 and almost surely for all $t \geq 0$.
- Thus, $\lim_{\varepsilon\downarrow 0} \zeta_t^H(\varepsilon)$ exists in L^2 and almost surely $\Longleftrightarrow$ the limit

$$\lim_{\varepsilon\downarrow 0}\,[g(\varepsilon) - g(-\varepsilon)] = \eta$$

  is finite.

Example (3)

Let $f(x) = \frac{1}{(x)_+^{1+\alpha}}$ with $0 < \alpha < \frac{1-H}{2H} \wedge \frac{1}{2}$. We have that $g(x) = \frac{1}{-\alpha}x^{-\alpha}$,

$$\lim_{\varepsilon\downarrow 0} \zeta_t^H(\varepsilon) = -\frac{1}{\alpha}\lim_{\varepsilon\downarrow 0} \varepsilon^{-\alpha}\mathscr{L}^H(\varepsilon, t) = -\infty$$

in L^2 and almost surely and

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left(\int_0^t 1_{\{B_s^H > \varepsilon\}}\frac{1}{(B_s^H)^{1+\alpha}}ds^{2H} - \frac{1}{\alpha}\varepsilon^{-\alpha}\mathscr{L}^H(0,t)\right)\\
&= \lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty \frac{\mathscr{L}^H(x,t) - \mathscr{L}^H(0,t)}{x^{1+\alpha}}dx\\
&= \text{p.v.}\int_{\mathbb{R}} \frac{\mathscr{L}^H(x,t)}{(x)_+^{1+\alpha}}dx
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exists in probability, this limit is called the (Cauchy) principal value (Hadamard's finite part) of the integral $\int_0^t f(B_s^H)ds^{2H}$, and it also is denoted by

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$$C_t(a) := \text{p.v.} \int_0^t \frac{ds}{B_s - a} := \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|B_s - a| > \varepsilon\}} \frac{ds}{B_s - a}.$$

and showed that $\frac{1}{\pi} C_t(\cdot)$ coincides with Hilbert transform of Brownian local time.

- T. Yamada (1984, 1985, 1986) also considered $C_t(a)$ and the following special case:

$$S_t^\alpha(a) = \text{p.v.} \int_0^t \frac{ds}{(B_s - a)_+^{1+\alpha}}$$
$$:= \lim_{\varepsilon \downarrow 0} \left\{ \int_0^t 1_{[a+\varepsilon,\infty)}(B_s) \frac{ds}{(B_s - a)^{1+\alpha}} - \alpha^{-1}\varepsilon^{-\alpha}\mathscr{L}(a,t) \right\}$$

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$\Longrightarrow$ Yamada's formulas:

$$B_t \log|B_t| - B_t = \int_0^t \log|B_s| dB_s + \frac{1}{2}C_t(0),$$

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$\Longrightarrow$ occupation times formulas:

$$\int_{\mathbb{R}} \mathcal{S}_t^\alpha(a) f(a) da = \alpha^{-1}\Gamma(1-\alpha) \int_0^t (\mathscr{D}_+^\alpha f)(B_s) ds,$$

$$\int_{\mathbb{R}} C_t(x) g(x) dx = \pi \int_0^t (\mathscr{H} g)(B_s) ds.$$

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$$P\left(C_t(0) \in dx\right) = \sqrt{\frac{2}{\pi^3 t}} \sum_{n=0}^{\infty} (-1)^n \exp\left\{-\frac{(2n+1)^2}{8t} x^2\right\} dx.$$

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  $\Longrightarrow$ Some limits associated with the following functionals:

$$F_0(t) = \mathcal{L}^H(0,t), \quad F_1(t) = \sup_x \mathcal{L}^H(x,t)$$

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  $\Longrightarrow$ Chaotic expansions of the processes

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with $0 < \alpha < \frac{1-H}{2H} \wedge \frac{1}{2}$ and introduced the fractional Yamada formulas:

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and

$$(B_t^H - x) \log |B_t^H - x| - (B_t^H - x) = -x \log |x| + x + \int_0^t \log |B_s^H - x| dB_s^H + \frac{1}{2} C_t^H(x).$$

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  - $H = \frac{1}{2}$: T. Yamada (1984).

- As mentioned before, in this talk, we consider the limit

$$\mathcal{K}_t^{H,f} := \lim_{\varepsilon\downarrow 0}\left(\int_0^t f(B_s^H)1_{\{|B_s^H|>\varepsilon\}}ds^{2H} + \zeta_t^H(\varepsilon)\right), \quad t\geq 0$$

in L^2 and almost surely, where f is not locally integrable and

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- Our objects are
  (1) to give the condition of existence;
  (2) to introduce an extension of Itô's formula by using the limit.
- Denote

$$G_+(x) = \int_x^M f(y)dy, \quad x>0$$

and

$$G_-(x) = \int_{-M}^x f(y)dy, \quad x<0$$

for some $M>0$.

- $B^H = \{B_t^H : t \geq 0\}$: a fractional Brownian motion with Hurst index $H \in (0,1)$, if it is a mean zero Gaussian process with $B_0^H = 0$ such that

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- B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from classical Itô analysis of Brownian motion are not available when dealing with B^H.
- B^H admits some interesting properties: we omit them.

- $\mathcal{H}$: the reproducing kernel Hilbert space of fBm.

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- When $\frac{1}{2} < H < 1$ we have

$$\mathcal{H} = \{\varphi : [0,T] \to \mathbb{R} \ | \ \|\varphi\|_{\mathcal{H}} < \infty\},$$

where

$$\|\varphi\|_{\mathcal{H}}^2 := H(2H-1)\int_0^T \int_0^T \varphi(s)\varphi(r)|s-r|^{2H-2} ds dr.$$

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- When $\frac{1}{2} < H < 1$, we usually use the subspace

$$|\mathcal{H}| = \left\{\varphi : [0,T] \to \mathbb{R} \mid \|\varphi\|_{|\mathcal{H}|} < \infty\right\}$$

  with

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- The divergence δ^H is the adjoint of D^H.
- $\mathbb{D}^{1,2} \subset \mathrm{Dom}(\delta^H)$;
- For an adapted process u, we denote $\int_0^t u_s dB_s^H = \delta^H(u 1_{[0,t]})$ (the fractional Itô integral).

- When $u \in \mathbb{D}^{1,2}$ and $\frac{1}{2} < H < 1$, we have

$$E\left[\left(\int_0^T u_s dB_s^H\right)^2\right]$$
$$= E\|u\|_{\mathcal{H}}^2 + \alpha_H^2 E \int_{[0,T]^4} D_\xi^H u_r D_\eta^H u_s (|\eta - r||\xi - s|)^{2H-2} ds dr d\xi d\eta$$

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- This is a main motivation studying principal values associated with fractional Brownian motion.

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- Then, the occupation formula implies that

$$\int_0^t f(B_s^H) 1_{\{|B_s^H|>\varepsilon\}} ds^{2H} = \int_\varepsilon^\infty \left[f(x)\mathscr{L}^H(t,x) + f(-x)\mathscr{L}^H(t,-x) \right] dx$$

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- Thus, the assumption $\int_{-M}^M |f(x)|dx = \infty$ for some $M > 0 \Longrightarrow$
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- Thus, the assumption $\int_{-M}^M |f(x)|dx = \infty$ for some $M > 0 \Longrightarrow$
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  - $f(x), f(-x) \not\equiv 0$ for $x > 0$.
- Thus, we get the following results.

Theorem (1)

Let f be continuous on $\mathbb{R} \setminus \{0\}$ such that $\int_{-N}^{N} |f(x)|dx = \infty$ for some $N > 0$. Then the limit

$$\lim_{\varepsilon \downarrow 0} \int_0^t f(B_s^H) 1_{\{|B_s^H| > \varepsilon\}} ds^{2H}$$

exists in probability (in L^2) if and only if the following conditions are satisfied:

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$$\lim_{\varepsilon \downarrow 0} \int_{-M}^{M} f(x) 1_{\{|x| > \varepsilon\}} dx;$$

(ii) *for some $M > 0$, the following convergence hold:*

$$\int_0^M G_+^2(x)dx, \quad \int_{-M}^0 G_-^2(x)dx < \infty,$$

and

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\frac{1-H}{2H}} G_+(\varepsilon) = 0, \quad \lim_{\varepsilon \downarrow 0} \varepsilon^{\frac{1-H}{2H}} G_-(-\varepsilon) = 0.$$

Theorem (2)

Let f be continuous on $\mathbb{R} \setminus \{0\}$ such that $\int_{-N}^{N} |f(x)|dx = \infty$ for some $N > 0$. Then the limit

$$\lim_{\varepsilon \downarrow 0} \int_0^t f(B_s^H) 1_{\{|B_s^H| > \varepsilon\}} ds^{2H}$$

exists almost surely if and only if conditions (i)-(ii) in Theorem 1 are satisfied and

$$\int_0^M \frac{dx}{x} \exp\left\{ -\frac{\alpha x}{\sup_{\{0<y\leq x\}} y^2 G_+(y)^2} \right\} < \infty,$$

and

$$\int_{-M}^0 \frac{dx}{|x|} \exp\left\{ -\frac{\alpha |x|}{\sup_{\{x\leq y<0\}} y^2 G_-(y)^2} \right\} < \infty$$

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Corollary (1)

Let f be continuous on $\mathbb{R} \setminus \{0\}$ such that $\int_{-N}^{N} |f(x)|dx = \infty$ for some $N > 0$. If (i) and (ii) in Theorem 1 hold, then the limit

$$\lim_{\varepsilon \downarrow 0} \left(\int_0^t f(B_s^H) 1_{\{|B_s^H| > \varepsilon\}} ds^{2H} + \zeta_t^H(\varepsilon) \right)$$

exists in probability if and only if $g(\varepsilon) - g(-\varepsilon)$ converges to a constant as $\varepsilon \downarrow 0$.

Corollary (2)

Let f be continuous on $\mathbb{R} \setminus \{0\}$ such that $\int_{-N}^{N} |f(x)|dx = \infty$ for some $N > 0$. Assume that (i) in Theorem 1 is false and that (ii) in Theorem 1 is true, then the limit

$$\text{p.v} \int_0^t f(B_s^H) ds^{2H} = \lim_{\varepsilon \downarrow 0} \left(\int_0^t f(B_s^H) 1_{\{|B_s^H| > \varepsilon\}} ds^{2H} + g(\varepsilon) - g(-\varepsilon) \right)$$

exists in probability (in L^2).

Example (5)

Taking $f(x) = \frac{1}{(x_+)^{1+\alpha}}$ with $0 < \alpha < \frac{1-H}{2H} \wedge \frac{1}{2}$, we see that

$$\lim_{\varepsilon \downarrow 0} \left(\int_0^t f(B_s^H) 1_{\{|B_s^H| > \varepsilon\}} ds^{2H} + \zeta_t^H(\varepsilon) \right)$$
$$= \lim_{\varepsilon \downarrow 0} \left(\int_0^t \frac{1}{(B_s^H)^{1+\alpha}} 1_{\{B_s^H > \varepsilon\}} ds^{2H} - \alpha^{-1} \varepsilon^{-\alpha} \mathscr{L}^H(0,t) \right)$$

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- By using the obtained results we give an Itô formula including the principal value.

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Let $0 < H < 1$ and let F be an absolutely continuous function on $\mathbb{R}$ such that F' is absolutely continuous on $\mathbb{R} \setminus \{0\}$. Suppose that

then, we have

$$F(B_t^H) = F(0) + \int_0^t F'(B_s^H)dB_s^H + \frac{1}{2}\beta\mathscr{L}^H(0,t) + \frac{1}{2}\text{p.v.}\int_0^t F''(B_s^H)ds^{2H}. \qquad (0.1)$$

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- Remark:

$$\text{p.v.}\int_0^t F''(B_s^H) ds^{2H} = \lim_{\varepsilon \downarrow 0} \int_0^t F''(B_s^H) 1_{\{|B_s^H| > \varepsilon\}} ds^{2H}$$

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(4)

$$F''(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 0, \\ \frac{1}{x} + \sin x, & \text{if } x < 0, \end{cases}$$

$\Longrightarrow \beta = 1$.

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(1) $F'(\varepsilon) - F'(-\varepsilon) \longrightarrow \infty$ *as* $\varepsilon \downarrow 0$;

(2) *the condition (i) in Theorem 1 with* $f = F''$ *is false;*

then, we have

$$F(B_t^H) = F(0) + \int_0^t F'(B_s^H)dB_s^H + \frac{1}{2}\text{p.v} \int_0^t F''(B_s^H)ds^{2H}. \qquad (0.2)$$

Theorem (4)

Let $0 < H < 1$ and let F be an absolutely continuous function on $\mathbb{R}$ such that F' is absolutely continuous on $\mathbb{R} \setminus \{0\}$. Suppose that

(1) $F'(\varepsilon) - F'(-\varepsilon) \longrightarrow \infty$ *as* $\varepsilon \downarrow 0$;

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(3) *the condition (ii) in Theorem 1 with* $f = F''$ *is true,*

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- Remark:

$$\text{p.v}\int_0^t F''(B_s^H)ds^{2H} = \lim_{\varepsilon\downarrow 0}\left(\int_0^t F''(B_s^H)1_{\{|B_s^H|>\varepsilon\}}ds^{2H} + F'(\varepsilon) - F'(-\varepsilon)\right).$$

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- L^2: the reproducing kernel Hilbert space of Brownian motion $\Longrightarrow$ the following Hardy type inequality.

Lemma (An extension of Hardy's inequality)

Let $\frac{1}{2} < H < 1$. *Then we have*

$$||\mathbb{H}f||_{|\mathcal{H}|} \le C_H ||f||_{|\mathcal{H}|}$$

for all $f \in |\mathcal{H}|$. *Moreover, when* $0 < H \le \frac{1}{2}$ *we have*

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- By using the inequality, we introduce the next convergence.

Proposition (1)

When $\frac{1}{2} < H < 1$, *the convergence*

$$\lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{1} f(s) B_s^H \frac{ds}{s} = \int_0^1 \mathbb{H} f(s) dB_s^H$$

exists in L^2 *and almost surely for all* $f \in |\mathcal{H}|$. *Moreover, when* $0 < H \leq \frac{1}{2}$, *the above convergence also holds for all* $f \in \mathcal{H}$.

Proposition (2)

When $\frac{1}{2} < H < 1$, *the convergence*

$$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 f(y) \frac{dy}{y} \int_0^t 1_{[|B_s^H| \le y]} dB_s^H = \int_0^t 1_{(|B_s^H| \le 1)} \mathbb{H} f\left(|B_s^H|\right) dB_s^H$$

exists in L^2 *and almost surely for all* $f \in |\mathcal{H}|$. *Moreover, when* $0 < H \le \frac{1}{2}$, *the above convergence also holds for all* $f \in \mathcal{H}$.

Corollary

For $\frac{1}{2} < H < 1$, *the convergence*

$$\lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{t} \left(\frac{B_s^H}{s} \right)^2 ds$$

exists in L^2 *and almost surely, for* $0 < H \leq \frac{1}{2}$, *the above limit does not exist in probability.*

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THANKS!